

REGULARITY IN WEIGHTED GRAPHS A SYMMETRIC FUNCTION APPROACH

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ABSTRACT. In this note we consider k -regular multigraphs, where the possible edge multiplicities are controlled. These structures are considered in a question recently posed by Brendan McKay. We express the generating functions using the scalar product of symmetric functions, and consequently give conditions on when the classes are D-finite. We appeal to symmetric species results of Mendez to write the expressions in a systematic way.

1. INTRODUCTION

The asymptotic enumeration of regular graphs is a compelling topic that has appeared in many forms in combinatorics over the past half century. There are several approaches, and each has its own conditions, and results. In this note we revisit the symmetric function approach, introduced by Goulden, Jackson and Reilly [7], generalized by Gessel [6], and automated by Chyzak, Mishna and Salvy [3]. The goal of this work is to give insight on a recent related problem posed by McKay:

Problem (McKay [12]). Let \mathcal{J} , and \mathcal{J}^* be subsets of the non-negative integers, and let $\mathbf{d} = d(n) = (d_1, \dots, d_n)$ be a vector of non-negative integers. Let $M(n, \mathcal{J}, \mathcal{J}^*)$ be the number of symmetric matrices whose diagonal entries are drawn from \mathcal{J}^* and off-diagonal entries from \mathcal{J} , whose row sums are d_1, \dots, d_n . As usual in graph theory, entries on the diagonal are counted twice. We are interested in the asymptotic value of $M(n, \mathcal{J}, \mathcal{J}^*)$ in the sparse case, where the row sums do not grow very quickly with n .

We consider this problem in the case that the d_i take on a finite number of values. (We call this set of possible values \mathcal{K}). We show that the generating function of the sequence $M(n, \mathcal{J}, \mathcal{J}^*)$ is D-finite under certain conditions.

A univariate generating function is D-finite if it satisfies a linear differential equation with polynomial coefficients. This property has considerable implications on asymptotic enumeration: D-finite functions have restrictions on their asymptotic form; asymptotic information is encoded in the differential equation that it satisfies; they can be treated with a number of automated tools. Although we do not compute asymptotic formulas here, the fact that the generating functions are D-finite can be useful for precisely such a computation.

We cast this problem in graph theoretic language as follows in order to state our results precisely. Let \mathcal{J} and \mathcal{K} be sets of positive integers, and suppose additionally that \mathcal{K} is finite. Let $\mathcal{G}_{\mathcal{J}, \mathcal{K}, n}$ be the set of well-labelled graphs on n vertices where edge weights are from $\mathcal{J} \cup \{0\}$ and the sum of weights of the edges incident to any given vertex is an element of \mathcal{K} . The case when all $d_i = k$ is the case of k -regular graphs. Here a graph is well-labelled if the label set is $\{1, 2, \dots, n\}$, where n is the number of vertices.

The following is our main result. It appears below as Theorem 8.

Theorem. *Let \mathcal{J} and \mathcal{K} be finite sets of positive integers. Let $G_{\mathcal{J}, \mathcal{K}}(z)$ be the generating function for the class $\mathcal{G}_{\mathcal{J}, \mathcal{K}}$ of well-labelled graphs where edge weights are from $\mathcal{J} \cup \{0\}$ and the sum of weights of the edges incident to any given vertex is an element of \mathcal{K} . Then,*

$$G_{\mathcal{J}, \mathcal{K}}(z) = \sum_n |\mathcal{G}_{\mathcal{J}, \mathcal{K}, n}| z^n$$

is D-finite. Furthermore, the differential equation satisfied by $G_{\mathcal{J}, \mathcal{K}}(z)$ is theoretically computable.

Key words and phrases. regular graphs, symmetric functions, D-finite, generating functions.

The condition of finiteness on \mathcal{J} is not necessary, but is a consequence of the finiteness of \mathcal{K} . We show how to relax the condition of finiteness, once there is more notation developed.

For example $G_{\{1\},\{k\}}(z)$ is generating function for simple, labelled, k -regular graphs, and $G_{\{1,2,\dots,k\},\{k\}}(z)$ is the generating function for labelled k -regular multigraphs. The D-finiteness of these generating functions for generic k was conjectured by Goulden, Jackson and Reilly [7], and was proved by Gessel [6]. Our strategy proves the more general result by using the work of Mendez [13] to create the same framework as of [6], to which the work of [3] then applies. This amounts to building a symmetric function encoding of the full class of graphs, and then performing a subseries extraction to realize the degree restriction. These are all theoretically effective, and hence the differential equations are potentially computable.

As Gessel noted, the class of all regular graphs is *not* D-finite. Consequently, our results, up to finite union of classes are most likely optimal.

1.1. Contribution: Structure of graphs with controlled edge multiplicities. McKay remarked in his problem “The simplest non-trivial case is $\mathcal{J}^* = \{0\}$ and $\mathcal{J} = \{0, 2, 3\}$ ”. In our notation, $\mathcal{J}^* = \{0\}$ corresponds to the criterion that the graphs have no loops, and our \mathcal{J} are the same. We first consider $d_i = k$ for all i , and loosen this to consider then $d_i \in \mathcal{K}$ for finite \mathcal{K} and conclude with some more general comments. We also address $\mathcal{J}^* = \{0, 1\}$.

Our principal contribution is a new formulation of his problem, and the resulting proof of D-finiteness. We find our approach to be of interest as it gives a new way to view McKay’s problem, and also a new example for the class of symmetric species. In this context, generalizations to hypergraph variants are very natural.

The outline of the strategy has three main steps:

- (1) Find a series encoding a superset of graphs with edge weights from \mathcal{J} ;
- (2) Rewrite this series as symmetric function using the power sum symmetric functions;
- (3) Extract the subseries of terms with the desired degree sequence via a scalar product operation.

In Section 2 we describe how to write the graph generating functions using symmetric functions. The construction generalizes our previous work [14] in a straightforward way using the multiassemblies studied by Mendez [13]. The construction uses the species theory formalism [2], but we leave the category theoretic details to previous sources to avoid a rather substantial detour that is well described elsewhere. We remark, however, this appears to be different than the description of graphs as a species recently developed by Gainer-Dewar and Gessel [5]. That said, our Proposition 4 bears some resemblance to the formulas in Section 6 of Henderson’s species generalization [8], which is the formalism upon which their work is based. Perhaps this problem is a good entry point for that theory.

The D-finiteness result follows quickly once we adapt Waring’s formula. This is explained in Section 4. We conclude with some directions on how to weaken the conditions as stated.

2. LABELLED GRAPH GENERATING FUNCTIONS

We start with a systematic encoding of graph classes using symmetric functions.

2.1. Simple graphs and X -generating functions. Let G be a simple graph with vertex set $V(G) = \{x_1, \dots, x_n\}$, and edge set $E(G)$. We associate to G the monomial $\pi(G)$ defined

$$\pi(G) = \prod_{\{x_i, x_j\} \in E(G)} x_i x_j = x_1^{d_1} x_2^{d_2} \dots x_k^{d_k},$$

where d_i is the degree of x_i . Let $\mathbf{G}(X)$ be the generating function of the set of all labelled simple graphs \mathcal{G} , each with a vertex set a subset of $X = \{x_1, x_2, \dots\}$:

$$\mathbf{G}(X) = \sum_{G \in \mathcal{G}} \pi(G) = \prod_{i < j} (1 + x_i x_j).$$

(Remark, this is a superset of the well-labelled graphs). To see the second equality, remark that every edge is either present once, or not at all. Similarly, if $\overline{\mathcal{G}}$ is the set of graphs that permit multiple edges (but not

loops),

$$\overline{\mathbf{G}} = \sum_{G \in \overline{\mathcal{G}}} \pi(G) = \prod_{i < j} \left(\frac{1}{1 - x_i x_j} \right),$$

as every edge exists some non-negative integer number of times.

Under this description, the set \mathcal{G} is a symmetric species [13], and the series encoding $\mathbf{G}(X)$ is what Mendez calls the associated X -generating function. Our strategy is to determine the X -generating function for the class of graphs of \mathcal{G} in which edge weights are incorporated, and treated as multiplicities. (Hence, that the edge weights be positive integers is essential.) It is straightforward to get an expression for this, and then it is a mechanical manipulation to get a form we desire.

The X -generating function for the class of labelled graphs with graphs edge weights from the set of positive integers \mathcal{J} is

$$(1) \quad \mathbf{G}_{\mathcal{J}}(X) = \sum_{G \in \mathcal{G}_{\mathcal{J}}} \pi(G) = \prod_{i < j} \left(1 + \sum_{s \in \mathcal{J}} (x_i x_j)^s \right).$$

For example, $\mathbf{G}_{\{1,2,3,\dots\}}(X) = \overline{\mathbf{G}}(X)$. When the X is clear, we might choose to not indicate the variable set.

To determine the series for graph classes with loops it is sufficient to change the product index from $i < j$ to $i \leq j$. If, rather, the loops have weights from a different set, say \mathcal{J}^* , we multiply Eq. (1) by the product $\prod_i (1 + \sum_{s \in \mathcal{J}^*} (x_i x_i)^s)$.

The number of well-labelled graphs in this class with given degree sequence d_1, \dots, d_n is simply the coefficient of the monomial $x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}$. For suitably defined graph classes, and degree vectors, this is the value under investigation by McKay. In standard generating function notation we write the coefficient as

$$[x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}] \mathbf{G}_{\mathcal{J}}.$$

As is always the case for symmetric species, the X -generating functions are symmetric functions. We access the coefficient using classic symmetric function operations.

2.2. Expressing $\mathbf{G}_{\mathcal{J}}(\mathbf{x})$ using symmetric functions. Since we can relabel any graph with a different set of labels and it remains in the class, $\mathcal{G}_{\mathcal{J}}$ is a symmetric class with respect to the graph labels. We leverage this underlying symmetry to rewrite the generating function in terms of symmetric functions.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be an integer partition of n , a fact which we denote by $\lambda \vdash n$. Let $X = x_1, x_2, \dots$ be an infinite, but countable, variable set. Then the symmetric function $m_{\lambda}(X)$ (or simply m_{λ}) is defined

$$m_{\lambda}(X) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_k}^{\lambda_k}$$

where the sum is over all k -tuples of distinct positive integers (i_1, i_2, \dots, i_k) . This is the monomial symmetric function indexed by λ . The set of the monomial symmetric functions form a basis for a vector space of symmetric functions over \mathbb{Q} .

We express the classic elementary, complete, and power sum symmetric functions in the monomial basis as follows:

$$e_n = m_{(1,1,\dots,1)}, \quad h_n = \sum_{\lambda \vdash n} m_{\lambda}, \quad p_n = m_{(n)}.$$

Recall the definition $e_{\lambda} = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}$. The set of the elementary symmetric functions indexed by partitions also forms a basis for a vector space of symmetric functions. This is also true for the sets of h_{λ} and p_{λ} respectively, which are similarly defined. We work in the ring of symmetric series over X :

$$\Lambda((X)) := \mathbb{Q}[[p_1, p_2, p_3, \dots]].$$

In particular we are interested in symmetric series $\mathbf{R}(X)$ of the form

$$\mathbf{R}(X) = \sum_{0 < n} \sum_{\lambda \vdash n} c_{\lambda} p_{\lambda}(X).$$

The symmetric function operation known as *plethysm* is essential to our solution. Given two symmetric functions u and v , the inner law defines the quantity $u[v]$ by defining the following rules, with $u, v, w \in \Lambda$ and α, β in K

$$(\alpha u + \beta v)[w] = \alpha u[w] + \beta v[w], \quad (uv)[w] = u[w]v[w],$$

and, most importantly, if $w = \sum_{\lambda} c_{\lambda} p_{\lambda}$ then $p_n[w] = \sum_{\lambda} c_{\lambda} p_{(n\lambda_1)} p_{(n\lambda_2)} \dots$. For example, we can deduce that $w[p_n] = p_n[w]$, and in particular that $p_n[p_m] = p_{nm}$. In a mnemonic way:

$$(2) \quad w[p_n] = w(p_{1n}, p_{2n}, \dots, p_{kn}, \dots) \quad \text{whenever} \quad w = w(p_1, p_2, \dots, p_k, \dots).$$

Let $\mathbf{H} = \sum_n h_n$ and $\mathbf{E} = \sum_n e_n$. Gessel noted that \mathbf{G} and $\overline{\mathbf{G}}$ can both be expressed using plethysm:

$$(3) \quad \mathbf{G} = \prod_{i < j} (1 + x_i x_j) = \sum_n e_n [e_2] = \mathbf{E}[e_2].$$

$$(4) \quad \overline{\mathbf{G}} = \prod_{i < j} \left(\frac{1}{1 - x_i x_j} \right) = \sum_n h_n [e_n] = \mathbf{H}[e_2].$$

Given Eq. (2), often plethysm expressions are easier to manipulate when the symmetric functions are written in the power sum basis. We do this next in Section 2.3, and this is followed by a discussion on how to derive the plethysm expressions for $\mathbf{G}_g(X)$ in 3.

2.3. Expressions in the power sum basis. We recall the following classic lemma as it guides our work. It shows how to express an infinite sum of h_n as a function of power sum symmetric functions.

Lemma 1 (Waring formula). *The following equations are true:*

$$\begin{aligned} \mathbf{H} &= \sum_n h_n = \prod_{i < j} \frac{1}{1 - x_i} = \exp \left(\sum_{0 < k} p_k / k \right), \\ \mathbf{E} &= \sum_n e_n = \prod_{i < j} (1 + x_i) = \exp \left(\sum_{0 < k} (-1)^{k+1} p_k / k \right). \end{aligned}$$

Proof. The proof is elementary series manipulations:

$$\log \prod_{0 < i} \frac{1}{1 - x_i} = \sum_{0 < i} \log \frac{1}{1 - x_i} = \sum_{0 < i} \sum_{0 < k} x_i^k / k = \sum_{0 < k} \sum_{0 < i} x_i^k / k = \sum_{0 < k} p_k / k.$$

□

Indeed, the plethysms are easier to analyse given this form:

$$\begin{aligned} \overline{\mathbf{G}} &= \mathbf{H}[e_2] = \exp \left(\sum_{0 < k} \frac{1}{k} p_k \right) [e_2] \\ &= \exp \left(\sum p_k [e_2] / k \right) \\ &= \exp \left(\sum \frac{1}{k} p_k [p_1^2 / 2 - p_2 / 2] \right) \\ &= \exp \left(\sum \frac{1}{2k} (p_k^2 - p_2 k) \right). \end{aligned}$$

We can similarly express $\mathbf{E}[e_2]$ (and indeed $\mathbf{H}[h_2]$, $\mathbf{H}[e_2]$).

3. LABELLED GRAPHS AS A SYMMETRIC SPECIES

One of the innovations of species theory [9, 2], is a rigorous combinatorial interpretation of the plethysm operation in terms of natural compositions of combinatorial structures [1]. Plethysm as an analytic analog to composition has been well studied since Pólya's composition theorem. Asymmetric series of Labelle [10] is also an important relative to the X -generating functions that we seek.

The combinatorial understanding of the composition in the particular case of \mathbf{H} and \mathbf{E} is formally developed by Mendez [13], and we gave a direct interpretation in the case of graphs, and other variants in [14]. In particular, [14] contains a description of graphs as multisort, and ultimately symmetric, species. The interpretation is as follows: a simple labelled graph is a set of edges. Edges are sets of atomic structures. Each atom is coloured a colour from the infinite set $X = \{x_1, x_2, \dots\}$, and atoms of the same colour are identified to form a vertex. The combinatorial composition of a set of edges is reflected in X -generating function by a plethysm.

In the notation of classic species, the class of all labelled multigraphs with loops allowed given by the multisort species $E[E_2](X)$, and the associated cycle index series is $\mathbf{H}[h_2]$. Here E is the species of sets, and E_2 is the species of a set of cardinality two. Thus, a graph is a set of edges, but repetitions are not controlled in the multisort version. The multiassembly construction of Mendez controls repetition of elements. A *multiassembly of type* λ , denoted M_λ , is a multiset where the multiplicities of the elements are prescribed by the parts of the partition λ . For example, a multiassembly of type $\lambda = (1, 1, \dots, 1)$ is a usual set without repetitions. Example 3.7 and Proposition 3.9 of [13] describe how to get the X -generating function associated a composition of a multiassembly and a some object. In particular, the X -generating function of a multiassembly $M_\lambda(X)$ is $m_\lambda(X)$, and the composition is realized by plethysm.

Example 2. We view graphs as multiassemblies of edges, where edge multiplicities are given by λ . The edges themselves are multiassemblies of vertices. For example, the species $M_{(3,2)}[M_{(1,1)}(X)]$ is the set of graphs with one edge of multiplicity two, and one edge of multiplicity three, under all possible positive integer labellings. There are two possible shapes, both under all possible valid labellings: The first graph is on three vertices, and is path. Each possible labelling contributes a monomial of the form $x_i^5 x_j^3 x_k^2$ to the X -generating function. The second shape is a graph with four vertices, and two edges, and the graph is not connected. Each graph of this type contributes a monomial $x_i^2 x_j^2 x_k^3 x_\ell^3$. The generating function is thus $m_{5,3,2} + m_{3,3,2,2}$. This is precisely $m_{(3,2)}[m_{(1,1)}]$.

The X -generating of $\mathbf{G}_\mathcal{J}$ of $\mathcal{G}_\mathcal{J}$ requires a sum over all possible partitions with parts from \mathcal{J} . Towards a more compact notation, define the symmetric function $f_{\mathcal{J},n}$ as follows:

$$(5) \quad f_{\mathcal{J},n} = \sum_{\lambda \vdash n; \lambda_i \in \mathcal{J}} m_\lambda.$$

To unravel the definition, remark $f_{\{2,3\},6} = m_{2,2,2} + m_{3,3}$ and further note that $f_{\{1\},n} = e_n$ and $f_{\{1,\dots,n\},n} = h_n$. We summarize these results in the following proposition.

Proposition 3. Fix \mathcal{J} , a nonempty integer partition, and $X = x_1, x_2, \dots$, and infinite, but countable set of labels.

- (1) The symmetric species of simple graphs $\mathcal{G}_\mathcal{J}$ is isomorphic to $\cup_{\lambda \vdash n; \lambda_i \in \mathcal{J}} M_\lambda[M_{(1,1)}(X)]$.
- (2) For any degree sequence $\mathbf{d} = (d_1, \dots, d_n)$ with sum $D = d_1 + \dots + d_n$, the coefficient

$$[x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}] \sum_{\lambda \vdash D; \lambda_i \in \mathcal{J}} m_\lambda[e_2]$$

is precisely the number of graphs in $\mathcal{G}_{\mathcal{J},\mathbb{N},n}$ with degree sequence \mathbf{d} . Equivalently, this is the coefficient of $m_{\mathbf{d}}$ when expanded in the multinomial basis of symmetric functions.¹

Similarly, the analogous class of graphs permitting loops², denoted $\mathcal{G}_\mathcal{J}^\circ$, is isomorphic as species to

$$\bigcup_{\lambda \vdash n; \lambda_i \in \mathcal{J}} M_\lambda[M_{(1,1)}(X) + M_{(2)}(X)],$$

and the coefficient

$$[x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}] \sum_{\lambda \vdash D; \lambda_i \in \mathcal{J}} m_\lambda[h_2]$$

is precisely the number of graphs in $\mathcal{G}_{\mathcal{J},\mathbb{N},n}^\circ$ with degree sequence \mathbf{d} .

3.1. Series expressions. To prove the D-finiteness results, we need the X -generating functions in a different format. We define $\mathbf{F}_\mathcal{J} = \sum_n f_{\mathcal{J},n}$ and hence $\mathbf{F}_{\{1\}} = \mathbf{E}$ and $\mathbf{F}_{\{1,2,\dots\}} = \mathbf{H}$. Now, $\mathbf{F}_\mathcal{J} = \prod_{0 < i} (1 + \sum_{s \in \mathcal{J}} x_i^s)$. We can generalize Lemma 1 to express $\mathbf{F}_\mathcal{J}$ in the power sum symmetric function basis. The proof follows from very basic manipulations.

¹With \mathbf{d} normalized to be decreasing if necessary.

²Graphs with loops of weight 1 corresponds to $\mathcal{J}^* = \{0, 1\}$ in the original problem.

Proposition 4. Let $\mathcal{J} = \{j_1, \dots, j_\ell\}$ be a set of distinct positive integers. Then,

$$\mathbf{F}_{\mathcal{J}} = \sum_{0 < n} f_{\mathcal{J},n} = \exp \left(\sum_{0 < n} a_n p_n \right)$$

where a_n is the following sum taken over all compositions $\alpha = (\alpha_1, \alpha_2, \dots)$ of n such that each part α_i is contained in \mathcal{J} :

$$(6) \quad a_n = - \sum_{\alpha} \frac{(-1)^{\text{parts}(\alpha)}}{\text{parts}(\alpha)}.$$

Here $\text{parts}(\alpha)$ is the number of parts in the composition.

Proof. First we note that as the elements of \mathcal{J} are all positive, a_n is well defined since the number of such compositions is finite. Next we apply the same log-exp expansion, and some very basic coefficient extraction formulas:

$$\begin{aligned} \log \sum_{0 < n} f_{\mathcal{J},n} &= \log \prod_{0 < i} (1 + \sum_{s \in \mathcal{J}} x_i^s) \\ &= - \sum_{0 < i} \log \frac{1}{1 - \sum_{s \in \mathcal{J}} (-x_i^s)} \\ &= \sum_{0 < i} \sum_{0 < k} \frac{(-1)^{k+1}}{k} \left(\sum_{s \in \mathcal{J}} x_i^s \right)^k \\ &= \sum_{0 < i} \sum_{0 < n} a_n x_i^n \\ &= \sum_{0 < n} a_n p_n. \end{aligned}$$

□

There are a few simplifications to note. If $\mathcal{J} = \{1\}$, then $a_n = \frac{-1}{n}$, since there is only one term in the summation in Equation (6). When $\mathcal{J} = \{1, 2, \dots\}$, $a_n = \frac{1}{n}$ as the sum is over all compositions, and we invoke a Möbius inversion argument. Further, when $\mathcal{J} = \{s_1, \dots, s_\ell\}$ is finite, we can express this as follows:

$$(7) \quad \sum_{0 < n} f_{\mathcal{J},n} = \exp \left(\sum_{0 < k} \frac{(-1)^{k+1}}{k} \sum_{i_1 + i_2 + \dots + i_\ell = k} \binom{k}{i_1 \ i_2 \ \dots \ i_\ell} p_{s_1 i_1 + s_2 i_2 + \dots + s_\ell i_\ell} \right).$$

Theorem 5. Let $\mathcal{J} = \{j_1, \dots, j_\ell\}$ be a finite set of ℓ distinct positive integers. Then $\mathbf{G}_{\mathcal{J}}$, the X -generating function for the symmetric species of labelled simple graphs with edge weights from \mathcal{J} satisfies:

$$\begin{aligned} \mathbf{G}_{\mathcal{J}} &= \left(\sum_n f_{\mathcal{J},n} [e_2] \right) = \mathbf{F}_{\mathcal{J}} [e_2] \\ &= \exp \left(\sum_{0 < n} \frac{a_n}{2} (p_n^2 - p_{2n}) \right), \end{aligned}$$

with a_n as defined in Proposition 4.

Proof. This follows from Propositions 3 and 4, and the fact that $e_2 = p_i^2 - p_2$. □

Returning to our example:

$$\mathbf{G}_{\{2,3\}} = \exp \left(\sum_{0 < n} \frac{(-1)^{n+1}}{2n} \sum_{i=0}^n \binom{n}{i} p_{3n-i}^2 - p_{6n-2i} \right).$$

4. D-FINITE SYMMETRIC SERIES

Recall that a series $S \in K[[x_1, \dots, x_n]]$ is *D-finite* in x_1, \dots, x_n when the set of all partial derivatives and their iterates, $\partial^{i_1+\dots+i_n} F / \partial x_1^{i_1} \dots \partial x_n^{i_n}$, spans a finite-dimensional vector space over the field $K(x_1, \dots, x_n)$. This was generalized to an infinite number of variables by Gessel [6], who had symmetric functions in mind. A series $S \in K[[x_1, x_2, \dots]]$ is *D-finite* in the x_i if the specialization to 0 of all but a finite (arbitrary) choice of the variable set results in a D-finite function (in the finite sense). In this case, many of the properties of the finite multivariate case hold true. One notable exception is closure under algebraic substitution, which requires additional hypotheses.

The definition is then tailored to symmetric series by considering the algebra of symmetric series as generated over \mathbb{Q} by the set of power sum symmetric functions $\{p_1, p_2, \dots\}$. A symmetric series is called *D-finite* when it is D-finite as a function of the p_i 's. The applicability of this definition will be apparent in a moment.

The two prototypical examples, **H** and **E** are easily seen to be D-finite, as any such specialization of variables results in an exponential of a polynomial, which is clearly D-finite. Similarly, from the expression in Proposition 4 we see that the same argument will hold for any $\mathbf{F}_{\mathcal{J}}$.

Theorem 6. *For any set of positive integers \mathcal{J} , the series $\mathbf{F}_{\mathcal{J}}$ and $\mathbf{G}_{\mathcal{J}}$ are both D-finite symmetric series with respect to the p -basis.*

Proof. For both of these symmetric series any specialization of the p variables so that only a finite number are non-zero leaves an exponential of a polynomial, which is easily shown to be D-finite in the remaining variables. We immediately conclude the D-finiteness of both $\mathbf{F}_{\mathcal{J}}$ and $\mathbf{G}_{\mathcal{J}}$, given the two previous results \square

4.1. Extracting the generating functions. In the power notation for integer partitions, $\lambda = 1^{n_1} \dots k^{n_k}$ indicates that i occurs n_i times in λ , for $i = 1, 2, \dots, k$. The normalization constant

$$z_{\lambda} := 1^{n_1} n_1! \dots k^{n_k} n_k!$$

plays the role of the square of a norm of p_{λ} in the following important formula:

$$(8) \quad \langle p_{\lambda}, p_{\mu} \rangle = \delta_{\lambda, \mu} z_{\lambda},$$

where $\delta_{\lambda, \mu}$ is 1 if $\lambda = \mu$ and 0 otherwise.

The scalar product is useful for coefficient extraction because $\langle m_{\lambda}, h_{\mu} \rangle = \delta_{\lambda, \mu}$. If we write S in the form $\sum_{\lambda} c_{\lambda} m_{\lambda}$, then the coefficient of $x_1^{\lambda_1} \dots x_k^{\lambda_k}$ in S is $c_{\lambda} = \langle S, h_{\lambda} \rangle$.

The closure under Hadamard product of D-finite series [11] yields the consequence:

Theorem 7 (Gessel [6]). *Let \mathbf{S} and \mathbf{T} be elements of $\mathbb{Q}[z][[p_1, p_2, \dots]]$, D-finite in the p_i 's and z , and further suppose that \mathbf{T} involves only finitely many of the p_i 's. Then $\langle \mathbf{S}, \mathbf{T} \rangle$ is D-finite as a function of z , provided it is well-defined as a power series.*

From this and Lemma 6 our main theorem follows almost immediately.

Theorem 8. *Let \mathcal{J} and \mathcal{K} be sets of positive integers, and suppose additionally that \mathcal{K} is finite. Let $G_{\mathcal{J}, \mathcal{K}}(z)$ be the generating function for the class $\mathcal{G}_{\mathcal{J}, \mathcal{K}}$ of well-labelled graphs where edge weights are from $\mathcal{J} \cup \{0\}$ and the sum of weights of the edges incident to any given vertex is an element of \mathcal{K} . Then,*

$$G_{\mathcal{J}, \mathcal{K}}(z) = \sum_n |\mathcal{G}_{\mathcal{J}, \mathcal{K}, n}| z^n$$

is D-finite. Furthermore, the differential equation satisfied by $G_{\mathcal{J}, \mathcal{K}}(z)$ is theoretically computable.

Proof. The series are combinatorial generating functions, and so they exist. We remark that

$$(9) \quad G_{\mathcal{J}, \mathcal{K}}(z) = \left\langle \mathbf{G}_{\mathcal{J}}, \frac{1}{1 - \sum_{k \in \mathcal{K}} h_k z^k} \right\rangle.$$

The first argument to the scalar product is D-finite by Lemma 6. Furthermore, $\sum_{k \in \mathcal{K}} h_k z^k$ is a polynomial in the power sum basis and z when \mathcal{K} is finite. Hence the second argument is rational, and D-finite. The result follows by Theorem 7. We address the computability in the next section. \square

Example 9. The series can be expanded:

$$\begin{aligned}\mathbf{G}_{\{3,2\}} &= \mathbf{F}_{\{3,2\}}[e_2] = (m_2 + m_3 + m_{2,2} + m_{3,2} + m_{2,2,2} + m_{3,3} + \dots)[e_2] \\ &= m_{2,2} + m_{3,3} + m_{4,2,2} + 3m_{2,2,2,2} + m_{5,3,2} + m_{3,3,2,2} \\ &\quad + m_{6,2,2,2} + 2m_{4,4,2,2} + 6m_{4,2,2,2,2} + m_{4,4,4} + m_{6,3,3} + 15m_{2,2,2,2,2,2} + 3m_{3,3,3,3} + \dots\end{aligned}$$

To extract the generating function of 2-regular graphs examine the coefficients of $m_{(2,2,\dots,2)}$:

$$G_{\{3,2\},\{2\}}(z) = \left\langle \mathbf{G}_{\{3,2\}}, \sum_{0 < n} h_2^n z^n \right\rangle = z^2 + 3z^4 + 15z^6 + 105z^8 + \dots$$

In this case, it corresponds to the number of matchings. Similarly,

$$G_{\{3,2\},\{2,3\}}(z) = \left\langle \mathbf{G}_{\{3,2\}}, \sum_{0 < n} \sum_{k=0}^n h_3^k h_2^{n-k} z^n \right\rangle = 2z^2 + 7z^4 + 36z^6 + 429z^8 + \dots$$

An extraction for any degree sequence is possible. The D-finiteness result can be generalized to handle infinite \mathcal{K} provided $\sum_n \sum_{\lambda \vdash n: \lambda_i \in \mathcal{K}} h_\lambda t^n$ is a D-finite symmetric series. In general, to determine the generating function of graphs with a fixed set of degree sequences \mathcal{D} , it suffices to consider the series $\sum_{\mathbf{d} \in \mathcal{D}} h_{\mathbf{d}} z^{\text{parts}(\mathbf{d})}$.

We could also mark something other than number of vertices. As we noted, the symmetric series $\sum h_n z^n$ is D-finite, and in fact, for any finite k , the series $\sum h_n^k z^n$ is D-finite. This would extract the generating function for the subclass of all regular graphs on k vertices from a given graph class with n marking the regularity. The resulting generating function is also D-finite.

4.2. Comments on effective computation. There are two computational tools at hand to compute $G_{\mathcal{J},\mathcal{X}}(z)$. One could iteratively expand $f_{\mathcal{J},n}[e_2]$ in the monomial basis as we did in the previous example. It might be slightly more efficient to expand the exponential expression. In practice, we were able to get some small results with this strategy.

Alternatively, we can make use of the fact that Gessel's result is effective: given the system of differential equations satisfied by symmetric series \mathbf{F} and \mathbf{G} , there are algorithms [3] to compute the differential equation satisfied by the scalar product. (Of course, at least one of \mathbf{F} and \mathbf{G} must contain other variables). It is straightforward to define the system satisfied by $\mathbf{G}_{\mathcal{J}}$, since it is expressed as an exponential of a polynomial. Consequently in theory we can compute the differential equation satisfied by $G_{\mathcal{J},\mathcal{X}}(z)$. In practice, using current algorithms, the computations are too resource intensive to deliver results when the number of variables is more than 5. We were able to confirm the correctness in small cases.

5. OTHER GENERALIZATIONS

By playing with the inner series in the plethysm, we can enumerate other families of objects, such as hypergraphs, or cyclic coverings of sets. The details are essentially given in [14].

As mentioned above, we could be interested in other kinds of functions for the growth of \mathbf{d} . Other options should be possible to extract, provided the generating function for the extractor is D-finite.

A different future direction would be to try to adapt the approach of de Panafieu and Ramos [4] for multigraphs to the weighted edge versions.

ACKNOWLEDGEMENTS

The author gratefully acknowledges funding from NSERC Discovery Grant.

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